# NONASSOCIATIVE ALGEBRAS SATISFYING IDENTITIES OF DEGREE THREE(1)

BY

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A number of different authors have studied classes of nonassociative algebras or rings satisfying a multilinear identity in three variables (see Bibliography). In this paper we are interested in the structure of those rings which satisfy a multilinear identity in three variables, but which do not fall into certain of the well-known classes of such rings (the latter turn out to be the exceptional cases of the general theory). In order to rule out certain less interesting cases, we shall, however, find it convenient to restrict our attention to those identities which are satisfied by at least one algebra with unity element. We show in §1, that, up to quasi-equivalence, every such identity either implies one of a certain one-parameter class of identities, or implies one of seven other identities, all but one of which have been treated in the literature. Rings satisfying $x^2x = xx^2$ and an identity from our one-parameter class are treated in another paper appearing in this issue [6]. Some remarks on the remaining seven identities are to be found in §2.

1. Let $I$ be a homogeneous multilinear identity in the three variables $x, y, z$ over the field $F$. Then $I$ is given by

$$\begin{aligned}(1)\qquad &\beta_1 xy\cdot z + \beta_2 yz\cdot x + \beta_3 zx\cdot y - \beta_4 yx\cdot z - \beta_5 zy\cdot x - \beta_6 xz\cdot y \\ &\quad + \gamma_1 x\cdot yz + \gamma_2 y\cdot zx + \gamma_3 z\cdot xy - \gamma_4 y\cdot xz - \gamma_5 z\cdot yx - \gamma_6 x\cdot zy = 0\end{aligned}$$

for some $\beta_1, \cdots, \beta_6,\ \gamma_1, \cdots, \gamma_6 \in F$. Defining $(x, y, z) = xy\cdot z - x\cdot yz$ and $[x, y] = xy - yx$, we first prove

THEOREM 1. *Let $A$ be a (possibly infinite-dimensional) algebra with unity element* 1 *satisfying an identity of the form* (1). *Then $A$ satisfies either*

$$(2)\qquad (x, y, z) + (y, z, x) + (z, x, y) - (y, x, z) - (x, z, y) - (z, y, x) = 0$$

*or*

$$(3)\qquad \alpha_1(y, x, x) + \alpha_2(x, y, x) + \alpha_3(x, x, y) + \alpha_4[[y, x], x] = 0$$

*for some choice of the $\alpha_i$'s.*

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Since a commutative algebra satisfies (2), we may restrict ourselves to the noncommutative case in the proof of this theorem. Let us assume first that the identity (1) satisfied by $A$ vanishes identically when any two of the variables are set equal. Then setting $y = x$ in (1) gives the relations $\beta_1 = \beta_4, \beta_2 = \beta_6, \beta_3 = \beta_5, \gamma_1 = \gamma_4, \gamma_2 = \gamma_6, \gamma_3 = \gamma_5$. Similarly, setting $z = x$ in (1) gives $\beta_1 = \beta_5, \beta_2 = \beta_4, \beta_3 = \beta_6, \gamma_1 = \gamma_5, \gamma_2 = \gamma_4, \gamma_3 = \gamma_6$, and all these relations together imply that the $\beta_i$'s are all equal and that the $\gamma_i$'s are all equal. Replacing $z$ by the unity element of $A$ in (1) now gives $\beta_1(xy - yx) + \gamma_1(xy - yx) = 0$, which shows that $\beta_1 + \gamma_1 = 0$, since $A$ is not commutative. Thus (1) reduces to (2) in this case.

On the other hand, if (1) does not vanish when $z$ is set equal to $x$, then $A$ satisfies an identity of the form

$$\delta_1 yx \cdot x + \delta_2 xy \cdot x + \delta_3 x^2 y + \delta_4 yx^2 + \delta_5 x \cdot yx + \delta_6 x \cdot xy = 0. \tag{4}$$

Replacing $x$ by $x + 1$ in (4) and subtracting (4) yields

$$\begin{aligned} 2\delta_1 yx + \delta_1 y + \delta_2(xy + yx) + \delta_2 y + 2\delta_3 xy + \delta_3 y + 2\delta_4 yx \\ + \delta_4 y + \delta_5(xy + yx) + \delta_5 y + 2\delta_6 xy + \delta_6 y = 0, \end{aligned}$$

which leads to the relations $2\delta_1 + \delta_2 + 2\delta_4 + \delta_5 = 0$ and $\delta_2 + 2\delta_3 + \delta_5 + 2\delta_6 = 0$ using the fact that $A$ is not commutative. Writing (4) in the form

$$\begin{aligned} -\delta_4(yx \cdot x - yx^2) + (\delta_2 + \delta_1 + \delta_4)(xy \cdot x - x \cdot yx) + \delta_3(x^2 y - x \cdot xy) \\ + (\delta_4 + \delta_1)(yx \cdot x - x \cdot yx - xy \cdot x + x \cdot xy) \\ + (2\delta_1 + \delta_2 + 2\delta_4 + \delta_5)\left(x \cdot yx - \frac{1}{2} x \cdot xy\right) \\ + \frac{1}{2}(\delta_2 + 2\delta_3 + \delta_5 + 2\delta_6) x \cdot xy = 0 \end{aligned}$$

and using the relations just derived gives (3) for an appropriate choice of the $\alpha_i$'s in terms of the $\delta_i$'s.

Next, let us consider what happens to (3) when $A$ is changed by quasi-equivalence. We shall say that $A_0$ is quasi-equivalent to $A$ if $A_0$ has the same set of elements as $A$ under the same operation of addition and if the operation of multiplication "$\circ$" in $A_0$ is related to multiplication in $A$ by

$$xy = \lambda(x \circ y) + \mu(y \circ x), \tag{5}$$

where $\lambda$ and $\mu$ are fixed scalars such that $\lambda + \mu = 1$ and $\lambda \neq \mu$[2].

LEMMA 1. *If $A$ satisfies* (3) *and if $A_0$ is related to $A$ by* (5), *then $A_0$ satisfies*

$$\begin{aligned} (\alpha_1\lambda - \alpha_3\mu)(y,x,x)^\circ + \alpha_2(\lambda - \mu)(x,y,x)^\circ + (\alpha_3\lambda - \alpha_4\mu)(x,x,y)^\circ \\ + [(\alpha_3 - \alpha_1)\lambda\mu + \alpha_4(\lambda - \mu)^2][[y,x]^\circ, x]^\circ = 0. \end{aligned} \tag{6}$$

*Hence,* $\alpha_1-\alpha_3$ *is invariant under this quasi-equivalence,* $\alpha_1+\alpha_2+\alpha_3$ *and* $\alpha_1+\alpha_3-2\alpha_2$ *are multiplied by* $(\lambda-\mu)$, *and* $\alpha_1-\alpha_3+4\alpha_4$ *is multiplied by* $(\lambda-\mu)^2$.

We have put zero superscripts on the associators and commutators in (6) to indicate that they are in $A_\circ$ rather than in $A$. To derive (6) we compute that

$$\begin{aligned}(x,y,z) &= xy\cdot z - x\cdot yz = \lambda(x\circ y)\cdot z + \mu(y\circ x)\cdot z - \lambda x\cdot(y\circ z) - \mu x\cdot(z\circ y)\\ &= \lambda^2[(x\circ y)\circ z] + \lambda\mu[z\circ(x\circ y)] + \lambda\mu[(y\circ x)\circ z] + \mu^2[z\circ(y\circ x)]\\ &\quad - \lambda^2[x\circ(y\circ z)] - \lambda\mu[(y\circ z)\circ x] - \lambda\mu[x\circ(z\circ y)] - \mu^2[(z\circ y)\circ x]\\ &= \lambda^2(x,y,z)^\circ - \mu^2(z,y,x)^\circ + \lambda\mu[z\circ(x\circ y) + (y\circ x)\circ z - x\circ(z\circ y) - (y\circ z)\circ x].\end{aligned}$$

The three special cases of this equation that we need are:

$$\begin{aligned}(y,x,x) &= \lambda^2(y,x,x)^\circ - \mu^2(x,x,y)^\circ + \lambda\mu[x\circ(y\circ x) + (x\circ y)\circ x\\ &\quad - y\circ(x\circ x) - (x\circ x)\circ y]\\ &= (\lambda^2+\lambda\mu)(y,x,x)^\circ - (\mu^2+\lambda\mu)(x,x,y)^\circ\\ &\quad + \lambda\mu[x\circ(y\circ x)\ (x\circ y)\circ x - (y\circ x)\circ x - x\circ(x\circ y)]\\ &= \lambda(y,x,x)^\circ - \mu(x,x,y)^\circ - \lambda\mu[[y,x]^\circ,x]^\circ,\\ (x,y,x) &= \lambda^2(x,y,x)^\circ - \mu^2(x,y,x)^\circ = (\lambda-\mu)(x,y,x)^\circ,\\ (x,x,y) &= \lambda^2(x,x,y)^\circ - \mu^2(y,x,x)^\circ\\ &\quad + \lambda\mu[y\circ(x\circ x) + (x\circ x)\circ y - x\circ(y\circ x) - (x\circ y)\circ x]\\ &= (\lambda^2+\lambda\mu)(x,x,y)^\circ - (\mu^2+\lambda\mu)(y,x,x)^\circ\\ &\quad + \lambda\mu[(y\circ x)\circ x + x\circ(x\circ y) - x\circ(y\circ x) - (x\circ y)\circ x]\\ &= \lambda(x,x,y)^\circ - \mu(y,x,x)^\circ + \lambda\mu[[y,x]^\circ,x]^\circ.\end{aligned}$$

We also have $[x,y] = xy - yx = \lambda x\circ y + \mu y\circ x - \lambda y\circ x - \mu x\circ y = (\lambda-\mu)[x,y]^\circ$ which may be iterated to give $[[y,x],x] = (\lambda-\mu)^2[[y,x]^\circ,x]^\circ$. Substituting these expressions into (3) immediately yields (6). The second half of Lemma 1 may be easily verified from (6).

LEMMA 2. *Every algebra* $A$ *satisfying* (3) *with* $\alpha_1,\alpha_2,\alpha_3$ *not all equal satisfies an identity of the form* (3) *with* $\alpha_1+\alpha_2+\alpha_3=0$. *For characteristic not* 3, *A satisfies an identity of the form* (3) *with* $\alpha_1+\alpha_2+\alpha_3\neq 0$ *if and only if* $A$ *satisfies* $(x,x,x)=0$. *If* $A$ *satisfies* (3) *with* $\alpha_1+\alpha_2+\alpha_3\neq 0$ *and* $\alpha_1\neq\alpha_3$ *and if* $F$ *has characteristic not* 2 *or* 3, *then* $A$ *satisfies* $x^3x = x^2x^2 = xx^3$. *If, furthermore,* $\alpha_1+\alpha_3-2\alpha_2\neq 0$ *and if* $F$ *has characteristic not* 5, *then* $A$ *is power-associative.*

If $A$ satisfies (3) with $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$, we may set $y = x$ in (3) to get $(\alpha_1 + \alpha_2 + \alpha_3)(x,x,x) = 0$, or $(x,x,x) = 0$. Conversely, $(x,x,x) = 0$ may be partly linearized to get

$$(y,x,x) + (x,y,x) + (x,x,y) = 0, \tag{7}$$

which is an identity of the form (3) with $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$ for characteristic not 3. If $A$ satisfies (3) with $\alpha_1, \alpha_2, \alpha_3$ not all equal, either $\alpha_1 + \alpha_2 + \alpha_3 = 0$ already, or $A$ satisfies (7) and we may subtract an appropriate multiple of (7) from (3) to achieve $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

If $A$ satisfies (3) with $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$ and $\alpha_1 \neq \alpha_3$, then $A$ satisfies $(x,x,x) = 0$ as well as (3) with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Setting $y = x^2$ in (7) gives $x^3x - x^2x^2 + x^3x - xx^3 + x^2x^2 - xx^3 = 0$, or $x^3x = xx^3$. Then setting $y = x^2$ in (3) with $\alpha_1 + \alpha_2 + \alpha_3 = 0$ gives $\alpha_1(x^3x - x^2x^2) + \alpha_3(x^2x^2 - xx^3) = 0$, or $x^3x = x^2x^2$, since $\alpha_1 \neq \alpha_3$. If the field $F$ has characteristic zero, the power-associativity of $A$ follows from this. Otherwise, we only know that $A^+$ is power-associative [1]. Letting $x^i$ denote the $i$th power of $x$ in $A^+$, we shall prove that $x^{n-i}x^i = x^n$ for each $n$ and $i < n$ using induction on $n$. Replacing $x$ and $y$ in (3) by $x^i$ and $x^{n-2i}$ respectively gives

$$\alpha_1(x^{n-i}x^i - x^{n-2i}x^{2i}) + \alpha_2(x^{n-i}x^i - x^nx^{n-i}) + \alpha_3(x^{2i}x^{n-2i} - x^ix^{n-i}) = 0,$$

where $x^{n-2i}x^i = x^ix^{n-2i} = x^{n-i}$ and $x^ix^i = x^{2i}$ by the inductive hypothesis. In the case of (7), this simplifies to

$$2[x^{n-i}, x^i] = [x^{n-2i}, x^{2i}], \tag{8}$$

and with the assumption $\alpha_1 + \alpha_2 + \alpha_3 = 0$, it simplifies to

$$-\alpha_3x^{n-i}x^i + \alpha_1x^ix^{n-i} - \alpha_1x^{n-2i}x^{2i} + \alpha_3x^{2i}x^{n-2i} = 0. \tag{9}$$

Defining $z = [x^{n-i}, x^i]$, we get $x^{n-i}x^i = x^ix^{n-i} + z$ and $x^{2i}x^{n-2i} = x^{n-2i}x^{2i} - 2z$ from (8), and substituting these into (9) gives

$$-\alpha_3x^ix^{n-i} - \alpha_3z + \alpha_1x^ix^{n-i} - \alpha_1x^{n-2i}x^{2i} + \alpha_3x^{n-2i}x^{2i} - 2\alpha_3z = 0,$$

or $(\alpha_1 - \alpha_3)(x^ix^{n-1} - x^{n-2i}x^{2i}) = 3\alpha_3z$. But putting $x^ix^{n-i} = x^{n-i}x^i - z$ and $x^{n-2i}x^{2i} = x^{2i}x^{n-2i} + 2z$ in here gives

$$(\alpha_1 - \alpha_3)(x^{n-i}x^i - x^{2i}x^{n-2i}) = 3\alpha_1z, \tag{10}$$

and adding these two equations yields

$$(\alpha_1 - \alpha_3)(x^ix^{n-i} + x^{n-i}x^i - x^{n-2i}x^{2i} - x^{2i}x^{n-2i}) = 3(\alpha_1 + \alpha_3)z.$$

Since $A^+$ is power-associative, the left side of this vanishes, leading to $(\alpha_1 + \alpha_3)z = 0$. Now the hypothesis $\alpha_1 + \alpha_3 - 2\alpha_2 \neq 0$ remains valid if any multiple of (7) is subtracted from (3), and hence is still true when we assume $\alpha_1 + \alpha_2 + \alpha_3 = 0$, in which case it becomes $\alpha_1 + \alpha_3 \neq 0$. But, under this hypothesis, $(\alpha_1 + \alpha_3)z = 0$ leads to $z = [x^{n-i}, x^i] = 0$. Thus, $x^{n-i}x^i = x^i x^{n-i} = \frac{1}{2}(x^{n-i}x^i + x^i x^{n-i}) = x^n$, as was to be shown.

We are now in a position to prove

THEOREM 2. *Let $A$ be an algebra of characteristic not two satisfying an identity of the form* (3) *not implied by* (7). *If $\alpha_1 \neq \alpha_3$ and $\alpha_1 - \alpha_3 + 4\alpha_4 \neq 0$ then either $A$ or a quadratic scalar extension of $A$ is quasi-equivalent to an algebra satisfying*

$$\alpha(y,x,x) - (\alpha + 1)(x,y,x) + (x,x,y) = 0 \tag{11}$$

*for some scalar $\alpha \neq 1$. Otherwise, $A$ is quasi-equivalent to an algebra satisfying one of the following identities*:

$$(y,x,x) = (x,x,y) + \frac{1}{2}[[y,x],x], \quad (y,x,x) = (x,y,x) + \frac{1}{4}[[y,x],x],$$

$$[[y,x],x] = 0, \; (y,x,x) + (x,x,y) = 2(x,y,x),$$

$$(y,x,x) + (x,x,y) = 2(x,y,x) - [[y,x],x].$$

COROLLARY. *Let $A$ be an algebra satisfying an identity which is a linear combination of associators, but which is not implied by $(x,x,x) = 0$. Then $A$ satisfies $(y,x,x) = (x,y,x)$,* (2), *or* (11) *for some scalar $\alpha$ (possibly equal to* 1).

If $A$ is an algebra satisfying the hypothesis of this theorem, it follows from Lemma 2 that we may assume that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Let us investigate first under what conditions the last term of (3) may be made to vanish under quasi-equivalence. Thus we wish to find $\lambda$ such that $0 = (\alpha_3 - \alpha_1)\lambda\mu + \alpha_4(\lambda - \mu)^2 = (\alpha_3 - \alpha_1)\lambda(1 - \lambda) + \alpha_4(2\lambda - 1)^2 = (\alpha_1 - \alpha_3 + 4\alpha_4)(\lambda^2 - \lambda) + \alpha_4$. If $\alpha_1 \neq \alpha_3$ and $\alpha_1 - \alpha_3 + 4\alpha_4 \neq 0$, there exists a permissible value of $\lambda$ satisfying this condition after possibly performing an appropriate quadratic extension of the field. Then, in this case, $A$ or a quadratic scalar extension of $A$ is quasi-equivalent to an algebra satisfying (3) with $\alpha_4 = 0$, $\alpha_1 \neq \alpha_3$, and $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Note that the latter two conditions remain valid under quasi-equivalence by Lemma 1. If $\alpha_3 = 0$, this leads to $(y,x,x) = (x,y,x)$, and $A$ is anti-isomorphic (and thus quasi-equivalent) to an algebra satisfying (11) with $\alpha = 0$. Otherwise we may divide (3) by $\alpha_3$ and obtain (11) with $\alpha = \alpha_1 / \alpha_3$.

In case $\alpha_1 \neq \alpha_3$ and $\alpha_1 - \alpha_3 + 4\alpha_4 = 0$, we have $\alpha_3 = \alpha_1 + 4\alpha_4$ and $\alpha_2 = -\alpha_1 - \alpha_3 = -2\alpha_1 - 4\alpha_4$. If $2\alpha_1 + 4\alpha_4 = 0$, (3) is $(y,x,x) = (x,x,y) + \frac{1}{2}[[y,x],x]$. Otherwise, we may choose $\lambda$ so that the coefficient of the third term vanishes, or so that $(\alpha_1 + 4\alpha_4)\lambda - \alpha_1\mu = (2\alpha_1 + 4\alpha_4)\lambda - \alpha_1 = 0$. This leads to $(y,x,x) = (x,y,x) + \frac{1}{4}[[y,x],x]$, since the conditions $\alpha_1 \neq \alpha_3$, $\alpha_1 - \alpha_3 + 4\alpha_4 = 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$ are preserved under quasi-equivalence.

There remains the case $\alpha_1 = \alpha_3$. From $\alpha_1 + \alpha_2 + \alpha_3 = 0$ we get $\alpha_2 = -2\alpha_1$. If either $\alpha_1$ or $\alpha_4$ is zero, we get $[[y, x], x] = 0$ or $(y, x, x) + (x, x, y) = 2(x, y, x)$, respectively. Otherwise, we observe from (6) that, when $\alpha_1 = \alpha_3$, each of the first three coefficients in (3) is multiplied by $(\lambda - \mu)$ under quasi-equivalence, while $\alpha_4$ is multiplied by $(\lambda - \mu)^2$. Thus the ratio of $\alpha_1$ to $\alpha_4$ may be taken to be any nonzero element of $F$. Although it will be most convenient in §2 to have this ratio be 3, we have selected $\alpha_1 / \alpha_4 = 1$ for this theorem to avoid having to exclude characteristic 3. This gives $(y, x, x) + (x, x, y) = 2(x, y, x) - [[y, x], x]$.

In essence, the results of this section show that the study of multilinear identities in three variables may be reduced to the study of the infinite class (11) and seven exceptions, namely, (7), (2), and the five identities listed at the end of Theorem 2. None of these identities alone seems to be strong enough to imply much structure without further assumptions. However, in [6] it is shown that any identity of type (11) implies a good deal when $(x, x, x) = 0$ is also assumed. This is already known for the case $\alpha = -\frac{1}{2}$ $(\alpha = -2)$, when (11) and $(x, x, x) = 0$ together are equivalent to the left (right) alternative law.

2. Since the structure of rings satisfying $(x, x, x) = 0$ and (11) is treated elsewhere, we devote the rest of this paper to a brief discussion of the remaining cases mentioned in §1. We begin by proving a result on algebras which are quasi-equivalent to an algebra satisfying (11) only after a quadratic extension of the base field has been made. The proof of this result depends on the structure theory developed in [6].

THEOREM 3. *Let $A$ be a simple (possibly infinite-dimensional) algebra over a field $F$ of characteristic not 2 or 3 satisfying $(x, x, x) = 0$ and (3) for $\alpha_1 \neq \alpha_3$ and $\alpha_1 + \alpha_3 - 2\alpha_2 \neq 0$, and containing an idempotent $e$ not the unity element. Furthermore, suppose that neither $A$ nor a quadratic scalar extension of $A$ is quasi-equivalent to an algebra that is right alternative. Then $A$ is a noncommutative Jordan algebra.*

If $A$ is already quasi-equivalent to an algebra satisfying (11) without having to make a quadratic scalar extension then the result follows from [6, Theorem 6] and the fact that flexibility and Jordan admissibility are preserved under quasi-equivalence. Otherwise let $K$ be the quadratic extension field of $F$ such that the scalar extension $A_K$ of $A$ is quasi-equivalent to an algebra $A'$ satisfying (11) for some $\alpha$. Then we may regard $A_K$ as an algebra over $F$ containing $A$ and having twice the dimension of $A$. Hence any idempotent $e$ of $A$ is also an idempotent of $A_K$ and $A'$, and the set $L = A'_{10}A'_{01} + A'_{10} + A'_{01} + A'_{01}A'_{10}$ is an ideal of $A'$ by [6, Lemma 4]. But then the elements of $L$ also form an ideal of $A_K$, and $L \cap A$ will be an ideal of $A$. Since $A$ was assumed simple, either $L \cap A = 0$ or $L \cap A = A$.

Suppose first that $L \cap A = 0$. But considering the half-spaces with respect to $e$, we have $A_{1/2} \subset (A_K)_{1/2} = A'_{1/2} = A'_{10} + A'_{01} \subset L$ so that $A = A_1 + A_0$. Then the hypothesis $\alpha_1 + \alpha_3 - 2\alpha_2 \neq 0$ for (3) is equivalent to the hypothesis $\alpha \neq -1$ for (11) which allows us to conclude that $(A'_{11})^2 \subset A'_{11}$, $(A'_{00})^2 \subset A'_{00}$, and $A'_{11}A'_{00} = A'_{00}A'_{11} = 0$ from [6, Theorem 3]. It follows from this that $A_1^2 \subset A_1$, $A_0^2 \subset A_0$, and $A_1A_0 = A_0A_1 = 0$, and hence that $A = A_1 \oplus A_0$. But if $e$ is chosen not equal to the unity element of $A$, then $A_0 \neq 0$ and $A$ is not simple, contrary to hypothesis.

Hence we must have $L \cap A = A$, or $A \subset L$. This inclusion relation leads immediately to $A_K \subset L$, since $L$ is closed under multiplication by elements of $K$ and since $A$ generates $A_K$ under multiplication by elements of $K$. Therefore $A' = L$ and $A'_{ii} = A'_{ij}A'_{ji}$. We may now use [6, Theorem 4] and [6, Lemma 5] to conclude that $A'$ is alternative, and hence flexible and Jordan admissible. But then $A_K$ and $A$ are also flexible and Jordan admissible, since these properties are preserved under quasi-equivalence. Thus, A is a noncommutative Jordan algebra [12].

Next we take a brief look at the various identities mentioned in §1 which are not connected with the infinite class (11). We start by treating the two remaining cases of (3) where $\alpha_1 \neq \alpha_3$.

Lemma 5. *A ring $A$ of characteristic relatively prime to* 2 *and* 3 *satisfies* $(y,x,x) = (x,x,y) + \frac{1}{2}[[y,x],x]$ *if and only if $A^+$ is associative.*

After simplification this identity may be put in the form $(yx)x + x(yx) + (xy)x + x(xy) = 2yx^2 + 2x^2y$, which states that $A^+$ is alternative. But it is known that a commutative alternative ring of characteristic relatively prime to 3 is associative and the converse is trivial.

Lemma 6. *Let $A$ be a ring of characteristic relatively prime to* 2 *satisfying* $(x,x,x) = 0$ *and*

$$(y,x,x) = (x,y,x) + \frac{1}{4}[[y,x],x]. \tag{12}$$

*Then, for any idempotent $e$, $A = A_1 \oplus A_0$.*

Setting $y = x^2$ in (12) gives $xx^3 = x^2x^2$, and hence, $A = A_1 + A_{1/2} + A_0$ with respect to $e$. Suppose now that $y \in A_{1/2}$. Then setting $x = e$ in (12) and using $(y,e,e) = ye \cdot e - ye = -ey \cdot e$ gives $-ey \cdot e = ey \cdot e - e \cdot ye + \frac{1}{4}[ye \cdot e - e \cdot ye - ey \cdot e + e \cdot ey]$, or $0 = 7ey \cdot e - 5e \cdot ye + ye \cdot e + e \cdot ey$. Replacing $ye$ by $y - ey$ yields $0 = 7ey \cdot e - 5ey + 5e \cdot ey + ye - ey \cdot e + e \cdot ey = 6ey \cdot e - 5ey + 6e \cdot ey + y - ey$, or $6[ey - ey \cdot e - e \cdot ey] = y$. But then $6(ey)[I - R_e - L_e]^2 = y[I - R_e - L_e] = 0$, implying that $0 = 6(ey)[I - R_e - L_e] = y$. Thus $A$ is the additive direct sum of the subgroups $A_1$ and $A_0$, which are orthogonal by a well-known consequence of $(x,x,x) = 0$. To prove the lemma it therefore suffices to establish that $A_1$ and $A_0$ are subrings. But putting $x$, $y \in A_e(1)$ and $e$ in the linearization of (12) gives

$$4(yx \cdot e - yx) = 4(xy \cdot e - xy + yx - e \cdot yx) + (yx \cdot e - xy \cdot e - e \cdot yx + e \cdot xy),$$

or $yx[-3R_e - 5L_e + 8I] + xy[3R_e + L_e - 4I] = 0$. Setting $yx = s_1 + s_0$ and $xy = t_1 + t_0$ gives $8s_0 - 4t_0 = 0$, or $2s_0 = t_0$. By symmetry, $2t_0 = s_0$, giving $s_0 = t_0 = 0$ or $A_1^2 \subset A_1$. The proof that $A_0^2 \subset A_0$ is identical.

Consider next the identity

$$(y,x,x) + (x,x,y) = 2(x,y,x) - [[y,x],x]. \tag{13}$$

It will be recalled that in the proof of Theorem 2 we found that, up to quasi-equivalence, the coefficient of the double commutator in this equation may be chosen as any nonzero element of the field. Let us here assume characteristic not 3 and choose this coefficient as $-3$. Then subtracting (7) from this modified identity gives $3[[y,x],x] = 3(x,y,x)$, or $(yx)x + x(xy) = 2(xy)x$. This identity has been studied in some detail in [9]. Thus, when $(x,x,x) = 0$ is assumed, (13) gives nothing new except for characteristic 3.

Now linearizing (7) and adding to (2) gives $(x,y,z) + (y,z,x) + (z,x,y) = 0$, which has been employed in [5], [8], and [9]. And finally, $[[y,x],x] = 0$ has been used in [9] and [13]. Therefore with the exception of (12), the identities not of the form (11) are not basically new. Recalling that (7) and (11) with $\alpha = 1$ give the flexible law and with $\alpha = -\frac{1}{2}$ and $-2$ give the left and right alternative laws we may combine the above remarks with [6, Theorem 6] to get

THEOREM 4. *Let $A$ be a simple algebra of characteristic not 2 or 3 satisfying $(x,x,x) = 0$ and a homogeneous identity in three variables not implied by $(x,x,x) = 0$. Furthermore, let $A$ have a unity element 1 and an idempotent $e \neq 1$. Then one of the following statements holds*:

(i) *$A^+$ is associative.*

(ii) *$A$ satisfies one of the identities $(x,y,x) = 0$, $[[y,x],x] = 0$, or $(x,y,z) + (y,z,x) + (z,x,y) = 0$.*

(iii) *$A$ is quasi-equivalent to an algebra satisfying $(yx)x = x(xy) = 2(xy)x$.*

(iv) *Either $A$ or a quadratic scalar extension of $A$ is quasi-equivalent to an algebra satisfying either $(y,x,x) = 0$ or $(y,x,x) = (x,x,y)$.*

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